Q.E.D., QED

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1 Introduction

Quantum electrodynamics (QED), as part of the standard model of particle physics, stands as one of the major pillars of fundamental physics. The standard model is our best theory of the subatomic constituents of the universe, but is also widely regarded as merely an effective theory—an approximation to some more fundamental theory that would unify gravity with the atomic forces. For several decades, physicists have been hopeful that new physics will be discovered that falls outside the scope of the standard model, but to date it has proven sufficient for all new data from the Large Hadron Collider. Many high-energy physicists have instead turned their attention to cosmology as a way to test beyond standard model theories. The early universe opens a window to physics well beyond our current standard model due to the extremely hot and dense conditions thought to obtain in the early stages after the big bang; the late universe also seems to require new physics in the form of dark matter. However, the evidence one can gather from cosmology is highly mediated by our best theory of gravity and its model of the universe.

A second option for insight into physics beyond the standard model comes from precision tests within the standard model. Unlike the early universe, precision testing does not require energy scales that go well beyond our current best theories; instead, one looks for minute discrepancies between measured and predicted phenomena within the standard model, often at low energies. One hope is that precision tests of the standard model will begin to reveal discrepancies that cannot be resolved by factoring in more detail from known physics. Failure to reconcile precision measurements with the standard model also provides hints as to what phenomena will become crucial for testing future theories. Since we expect that the standard model will be succeeded by a new, more fundamental theory, it is useful to look to the details of the current generation of precision tests. We compare the current state-of-the-art in precision testing QED—measuring the anomalous magnetic moment of the electron—with

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the best known example of precision testing playing a significant role in theory change—Newtonian celestial mechanics. Using this comparison, we (1) outline the research program dictated by the structure of QED; (2) demonstrate the ways in which successful precision predictions relate to pure QED and QED as part of the standard model; and (3) outline how precision tests can eventually lead to discrepancies and hint at a new theory beyond the standard model. Most precision tests of QED can also serve as indirect measurements of the fine-structure constant α, since it is the fundamental coupling constant on which all QED predictions rely.

Smith (2014) has argued that research programs in physics generate knowledge by assuming a theoretical framework as a working hypothesis, then using it to search for the dominant causal factors governing a system. The theoretical framework allows physicists to determine, among other things, the values of fundamental theoretical parameters based on experiments and observations. Progress consists in resolving slight discrepancies among different results, reflected for example in disagreements in measured values of the parameters, by adding more detail into the model of the phenomenon while maintaining the essential correctness of the framework. This can be accomplished through (1) improving the mathematical tools of analysis; (2) creating more realistic models of the known causal factors; or (3) including sub-dominant causal factors that were neglected on the first analysis. Trust in the framework as guiding this process only breaks down when there is a persistent failure to resolve discrepancies. Even if the framework is superseded, it may have still correctly identified the dominant causal factors governing the phenomena within a restricted domain. Smith's work focuses on celestial mechanics, but as we will argue here, a similar account holds for precision tests of the standard model.

The remainder of this paper is organized as follows. In § 2 we outline some history regarding predictions in QED, and the structure of the relationship between theory and evidence. The relationship is one of ever-increasing precision in measurements of electromagnetic phenomena and their corresponding theoretically predicted values. Converging, sometimes independent lines of evidence have tightly constrained the low-energy value of the fine-structure constant over the past several decades. We distinguish between testing frameworks of pure QED ($\mathrm{QED_P}$) and QED as part of the standard model ($\mathrm{QED_{SM}}$). Next, in § 3, we detail the current state-of-the-art in experimental (§ 3.1) and theoretical (§ 3.2) tests of QED. Right now, the most precisely determined prediction is of the anomalous magnetic moment of the electron, though there are many other precision tests of QED. The theoretical prediction already signals a small discrepancy between a $\mathrm{QED_P}$ prediction and the measured value of the anomalous magnetic moment. In this case, however, the discrepancy is resolved within the standard model, and so no new physics is yet required. In § 3.3, we discuss a recent Cesium recoil experiment that is in statistically significant tension with the determination of α from a_e. New discrepancies lead to greater refinement of measurements, and could point the way to new physics if they persist. In § 4 we briefly discuss the quantum Hall effect, one of a few QED-independent methods for determining the fine-structure constant. Independent precision tests like these provide confidence that "closing the loop" through QED-mediated phenomena is not viciously circular. In § 5 we argue that this sort of precision testing could be highly useful in constructing models of physics beyond the standard model. The argument proceeds largely by analogy with the transition from Newtonian gravity to the theory of general relativity. Further, we argue that the knowledge generated in precision testing the standard model is robust to theory change. Even if the standard model is replaced, it has still generated genuine scientific knowledge, and the reasoning used to generate predictions remains reliable.

2 How predictions are made in QED

When one thinks about the relationship between theory and experiment in particle physics, quantum electrodynamics (QED) is often the paradigm example of close agreement and interplay between abstract formalism and experimental phenomena. Indeed, physicists routinely use superlatives in describing QED—as 'the most rigorously tested theory ever', or as having achieved 'the most precise agreement between theory and experiment in all of science'. Physicists bestow these accolades after QED has survived increasingly precise testing from the 1940s onward. The most famous test of QED is a determination of the anomalous magnetic moment of the electron.[1]

The electron's spin was discovered experimentally by Uhlenbeck and Goudsmit (1925), and the effort to include spin in a quantum mechanical description of the electron's interaction with the electromagnetic field led to Dirac's (1928) equation. According to Dirac's equation,the electron is a point particle, and its magnetic moment due to its intrinsic spin is given by

$$\mu_S = -g\frac{e}{2m_e}\mathbf{S} \tag{1}$$

where e is the charge of the electron, m_e its mass, $\mathbf{S}$ its spin, and the factor of g has the value of 2.

QED's initial triumph came shortly after it was written in covariant renormalized form, in the form of Schwinger's calculation of the first few significant figures in the anomalous magnetic moment of the electron, so called because it departs from Dirac's value: $a_e = (g-2)/2$.[2] A pair of experiments conducted in 1947-48, one by Nafe, Nelson, and Rabi (1947) via hyperfine splitting in hydrogen and deuterium, and the other by Kusch and Foley (1948) via Zeeman splitting in various elements, measured a value of $a_e(\text{experiment}) = 0.00119(5)$. Schwinger (1948) developed techniques to handle radiative corrections, taming the divergences that had plagued earlier attempts to calculate quantities such

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[1] Lautrup and Zinkernagel (1999) give a clear survey of different experimental techniques used to study the anomalous magnetic moment from roughly 1948-1995, and argue that this case does not involve a problematic form of theory-ladenness.

[2] See, in particular, Schweber (1994) for a detailed history of the development of QED up to 1948.

as the self-energy of the electron. Using these techniques, Scwhinger found a correction to Dirac's value for the electron's dipole moment to first order in the fine structure constant α, $a_e(\text{theory}) \approx 0.00162$. The result was calculated by renormalizing the one loop contribution to the electron vertex function. The value—which is inscribed on Schwinger's tombstone—was in agreement with the contemporary experimental results, and this important success led to the widespread acceptance of QED. Today, as we will discuss below, the precision of agreement between theoretical and experimental determinations of a_e extends to $\Delta a_e = 0.91 \times 10^{-12}$.

Central to this test—and all other precision tests—of QED is the fine-structure constant, $\alpha = e^2/\hbar c$, the dimensionless parameter characterizing the coupling strength between electrons, positrons, and photons. As QED was extended to become part of the standard model, α characterized more generally the vertex between massive charged leptons, their antiparticles, and photons. The fine structure constant was initially proposed by Sommerfeld (1921) in a relativistic extension of the Bohr model of the hydrogen atom. This was the motivation for introducing a relativistic spin quantum number for the electron; α was then interpreted as the ratio of the velocity of the electron in its lowest Bohr orbit to the speed of light. With the advent of QED in the 1940s, α was reinterpreted as the fundamental dimensionless coupling constant for pure electromagnetic systems (i.e., consisting of positrons, electrons, and photons only).

Famously, quantum field theories do not predict the numerical values of their coupling constants. After renormalization, the physical charge of the electron in QED is strictly an empirical input. So predictions of quantities like a_e depend on a measured value of α (or e). Additionally, predictions for most QED effects depend on a perturbative expansion of the generating functional, in powers of the coupling constant α.[3] An observable quantity F is expanded as a power series

$$F(\alpha) = \sum_{n=0}^{\infty} A_n \left(\frac{\alpha}{\pi}\right)^n , \tag{2}$$

where Feynman diagrams for the interaction are used to calculate the $\{A_n\}$ up to a given value of n. Predictions have constantly been improved by calculating effects to higher orders in the perturbative expansion.[4] This complicates the

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[3] Notable exceptions are calculations of bound states in QED, which are nonperturbative effects. In order to calculate bound states, the canonical formalism is often a better choice. See Kinoshita (1990, Chs. 12,13,15–17) for detailed discussions of bound states in QED.

[4] At some point, including more terms in the expansion will actually lead to decreased accuracy of predictions; the perturbative expansions in quantum field theory are actually divergent series, and are assumed to be asymptotic expansions of some unknown exact formal expression. Asymptotic expansions approximate a function up to some finite order, at which point additional terms take one further away from the true value of the function. To say that Equation (2) is asymptotic, we mean that the radius of convergence is zero; no matter how small α is, the limit

$$\lim_{N\to\infty} \sum_{n<N} A_n \left(\frac{\alpha}{\pi}\right)^n$$

will diverge. For QED, one expects that, for large enough n, the value of A_n will be of order picture of “confirming” QED beyond a simple hypothetico-deductive case of deriving a value of a_e then comparing to experiment; instead there is a continued process of refining the measured value of α, using it as input for calculating higher order perturbative expansions, and comparing more precise predictions to a new generation of precision experiments.

If the determination of a_e were the only QED effect to enter into this cycle, one might worry that the converging results were circular and therefore doing little to confirm QED. On our view, this kind of reasoning need not be viciously circular; evidence can accrue to the theory through a process of successive refinements. But for those who share this worry, there are many experimental effects which can, when combined with the theoretical apparatus of QED, be used to either determine the value of α or compare to the QED prediction. Importantly, there are also ways to measure α that do not depend on QED. These independent determinations of α come from effects in condensed matter physics, and provide a tight consistency check on the converging QED results. As one can see from Table 1, the various means for determining α show remarkable agreement to very high levels of precision. Many of these values are mediated by QED, though the phenomena vary. The most precise are the low-energy QED effects and the condensed matter measurements. The latter do not depend on QED to mediate between the observed effect and an corresponding inferred value of α. These results depend on effects that can be modelled with ordinary quantum mechanics and classical electromagnetism.

In describing the framework for precision tests of electromagnetic phenomena, we will employ a contrast between two ways of considering QED:[5]

> **Pure QED** ($\mathrm{QED_P}$): Relativistic quantum theory describing interactions among photons, electrons, and positrons, using only the QED interaction vertex, propagators, and Feynman rules; α is the coupling constant.
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> **QED as part of the Standard Model** ($\mathrm{QED_{SM}}$): Sector of the Standard Model, describing interactions among photons and leptons if electroweak symmetry is broken. Interactions in $\mathrm{QED_{SM}}$ will in general include weak propagators, residual effects from charged quarks or hadrons, and interactions with the Higgs boson.

Pure QED dictates a strategy for precision tests of electromagnetic phenomena: first, determine a simple leading order prediction of some effect, using the best available value of α. This is the $n = 1$ term from Eq.(2). Next, this value is compared to the best available measurements; if the two values agree within their respective uncertainties, one aims to improve precision on both sides thereby reducing the uncertainties. If a discrepancy exists between theory and experiment, one should first aim to improve the theoretical prediction, including any

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$n!$, which ensures that the above limit diverges for all values of α.

[5] This contrast is admittedly ahistorical, since much of the precision testing has been conducted since the development of the standard model. Yet calculations like those discussed below typically segregate QED effects from other standard model contributions, in effect giving what we call a $\mathrm{QED_P}$ result alongside the full $\mathrm{QED_{SM}}$ calculation.

Table 1: Determinations of α^{-1}. Low-energy QED values are taken from articles cited below. All other values are from Peskin and Schroeder (2018, p. 198). The most precise values come from low-energy QED tests and condensed matter. (This list is not comprehensive; in particular, we have not included several lower precision constraints from other scattering experiments.)

Low-energy QED	
e^- anomalous magnetic moment	137.035 999 149 1 (33 0)
Atom recoil measurement (Rb)	137.035 999 049 (90)
Atom recoil measurement (Cs)	137.035 999 046 (27)
Spectroscopic Measurements	
Neutron Compton wavelength	137.036 010 1 (5 4)
Muonium hyperfine splitting	137.035 994 (18)
Lamb shift	137.036 8 (7)
Hydrogen hyperfine splitting	137.036 0 (3)
Condensed Matter	
Quantum Hall effect	137.035 997 9 (3 2)
AC Josephson effect	137.035 977 0 (7 7)
Scattering	
Cross sections for e^+e^- reactions	136.5 (2.7)

additional details that may be relevant. In both cases, improved theoretical precision comes initially from including higher-order terms in the expansion.

Difficulties arise at higher orders for a few reasons. First, the number of Feynman diagrams included in the determination of A_n increases factorially with n. Second, each individual diagram contributes more and more complicated integrals at high n, meaning that numerical methods are needed to solve the integrals. Finally, there is good evidence that the series expansions diverge, and are therefore thought to be *asymptotic* expansions. Asymptotic expansions only provide good approximations up to some finite order, after which the prediction gets worse and worse. There is also no way to exactly determine the order n at which this occurs, without knowing the underlying function to which the expansion is asymptotic.[6] If it were possible, in practice, to calculate arbitrarily high order contributions to the expansion, at some point the theoretical prediction would diverge from the experimental value. This last issue is

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[6] Though there are heuristic arguments to give an order at which the expansion will diverge. For a given order of expansion n, one expects $\mathcal{O}(n)$ diagrams to contribute, while each diagram is suppressed by a factor of g^n, with g the coupling constant for the theory. The order at which the number of diagrams is approximately equal to the suppression is $n \approx g^{-1}$, around $n = 137$ for QED. We are grateful to David Wallace and an anonymous reviewer for pointing this out.

only one of principle, since the practical difficulty of determining higher-order contributions means that the state-of-the-art is an expansion up to α^5.

If discrepancies persist between QED_P predictions and experiment at high precision, then this suggests that new physics beyond QED_P may be relevant for the measured quantity. We *can* treat pure QED as a stand-alone theory without assuming that it is part of the standard model. Although this would be an incomplete theory (for various reasons), we can ask whether there is a domain of phenomena which it nonetheless describes at high precision without needing to include other interactions—these would be pure QED systems. Discrepancies with respect to such systems would indicate that QED does not fully describe even this restricted domain. In this case, the natural next step is to shift to a different testing framework: QED_SM. Within this framework, we regard QED_P as an incomplete idealization. It is natural to seek to resolve any discrepancies by including standard model effects left out of QED_P. Furthermore, given our knowledge of the standard model as a whole, we can see that a system of electrons, positrons, and photons is the closest we can get to a pure QED system.

When we shift from considering the testing framework of QED_P to QED_SM, the strong, weak, and electromagnetic interactions all become relevant, and all contribute to interparticle interactions in highly complex ways. In particular, the self-energy of the electron may include contributions from all allowed standard model interactions in principle, so some residual effects from the strong and weak sectors may play a noticeable role as precision of measurement increases. Though the process of testing is largely the same, new forces become important in the framework of QED_SM, and so new physics (from the perspective of QED_P) are now relevant. Finally, if one cannot account for discrepancies by including standard model effects, one has reason to believe that some new physics is contributing to the measured anomalous magnetic moment. We will see this process in action in the next section.

3 Precision measurements and determination of α

3.1 Measuring a_e

The first experimental measurement of the magnetic moment of the free electron, showing that it deviates from Dirac's value ($g = 2$), came from spectroscopic measurements of bound electrons. Kusch and Foley (1948) subjected beams of gallium, indium, or sodium atoms to an oscillating magnetic field, and determined the frequencies required to induce Zeeman splitting. These measured frequencies are related to the g-factor of the bound electron as $\hbar\omega = g\mu_b B_0$, where $\mu_b = \frac{e\hbar}{2m_e}$ is the Bohr magneton and B_0 is the magnetic field strength. They avoided the challenge of determining B_0 to high precision by considering the ratio of frequencies associated with different transitions. This experiment—and all subsequent studies of bound electrons—provide only indirect measurements of the anomalous magnetic moment of the electron. Bound electrons couple to external magnetic fields through their total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{s}$, which includes both spin ($\mathbf{s}$) and orbital ($\mathbf{L}$) angular momentum. Spectroscopic experiments measure the full g-factor (g_J), and further assumptions are needed to extract the value for the free electron. Kusch and Foley evaluated a_e based on a particular assumption about the electronic coupling. Uncertainties in the theoretical description of the spin-orbit coupling, and of the atomic nucleus itself, pose fundamental limits to precision in measurements of a_e from bound electrons.

Measurements of free electrons are a promising avenue to attain higher precision: in principle, such measurements could directly determine a_e, avoiding the complications arising from atomic binding and the beyond-QED physics governing the constituents of the atom. Bohr discouraged this idea in the early days of quantum mechanics. He argued against the viability of free-electron measurements of the anomalous magnetic moment using a Stern-Gerlach apparatus, on the grounds that the separation of an electron beam into distinct classical trajectories based on spin would violate the uncertainty principle (Garraway and Stenholm 2002). As Louisell, Pidd, and Crane (1954) emphasized, this line of argument was often taken to imply a much more sweeping prohibition of measurements based on free electrons than was warranted. They designed an experiment that determined the value of a_e based on the precession of electron spin as a beam passed through a uniform magnetic field, between two scatterings. (There is then a simultaneous measurement of one component of $\mathbf{S}$ and the position.) Within a decade, experiments based on spin precession in a static magnetic field eclipsed the precision attained by spectroscopic study of bound electrons (Rich and Wesley 1972).

The highest precision measurements achieved to date are based on what Dehmelt called a "geonium" atom: an electron held in a bound state by an external field. A device called a Penning trap effectively replaces the binding forces of an atomic nucleus with an adjustable combination of electromagnetic fields. This is a purely QED system, with the essential physics fully described in terms of leptons and interactions with the electromagnetic field. Stripping the nucleus out of the system removes the complications and sources of imprecision in atomic measurements, enabling incredibly high precision direct measurements of a_e.

A charged particle in a uniform magnetic field (with field strength B) moves in a circular cyclotron orbit, with a cyclotron frequency $\omega_c = \frac{q}{mc}B$. A Penning trap confines particles radially, roughly within a plane, using a uniform magnetic field along the z-axis. The particles are further prevented from moving away from the plane along the field lines by an electric quadrupole field, produced by three electrodes—two end caps shaped as hyperboloids, and one ring electrode. These features of the trap are illustrated in Figure 1. The addition of the electrostatic field modifies the simple cyclotron motion in two ways. First, the quadrupole potential confines the particles, and also leads to simple harmonic motion with a frequency ω_z along the z-axis. Second, the cyclotron frequency is reduced slightly (to ω_c') and the center of the cyclotron motion drifts. This

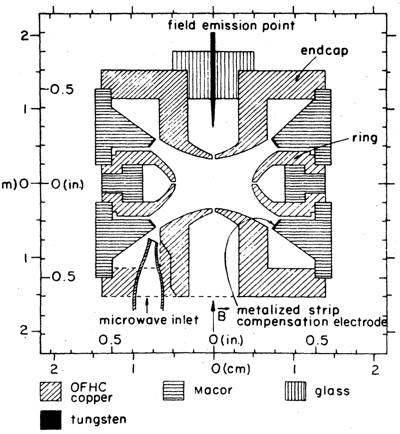

Figure 1: Scale drawing of a Penning trap, from L. S. Brown and Gabrielse (1986).

slow drift is called the magnetron motion and has a much lower frequency ω_m.

The energy levels of the geonium atom consist of cyclotron energy levels (Landau levels), with further line splitting for spin, axial harmonic motion, and the magnetron motion. For an electron moving in a uniform magnetic field, there is a spin precession frequency ω_s in addition to the cyclotron frequency ω_c. If these frequencies were identical, the energy levels of the atom would be degenerate: the $s = +1$ state at a given cyclotron level n would have the same energy as the $s = -1$ state at the cyclotron level $n + 1$. Yet due to the anomalous magnetic moment of the electron there is a small difference between these frequencies: $\omega_a = \omega_s - \omega_c$. The anomaly a_e can then be expressed in terms of this frequency, $a_e = \frac{\omega_a}{\omega_c}$.

Current measurements incorporate a number of ingenious experimental techniques for controlling the geonium atom and measuring the frequencies of transitions between different states to extremely high precision (see L. S. Brown and Gabrielse 1986). The axial resonance is particularly important, as it can be directly detected and experimentally manipulated by monitoring the voltage between the endcap and ring electrodes. The number of particles in the trap can be controlled, and a single particle can be maintained in a stable state for extremely long periods of time. Dehmelt (1990) famously kept one positron, which he named "Priscilla," in a Penning trap for 3 months. The coupling between spin and cyclotron motions to axial motion is enhanced by introducing an additional inhomogeneous magnetic field (a "magnetic bottle"). Dehmelt introduced a technique based on the "continuous Stern-Gerlach effect": the magnetic field induces a coupling between the axial motion of the electron and its spin orientation. Rather than the familiar spatial separation of different spin states, there is a separation in axial frequency, which is continuously monitored. Schematically, the experiment proceeds by driving the electron into a higher energy level,

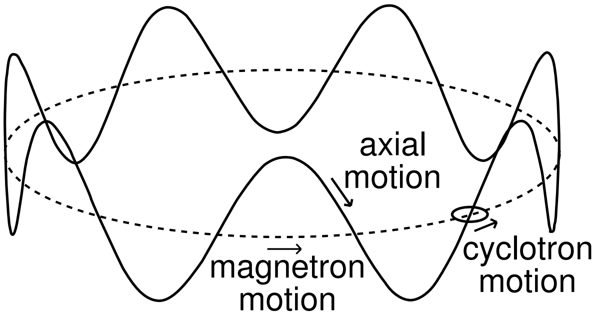

Figure 2: The motion of a particle in a Penning trap is a combination of three decoupled motions: magnetron motion at a frequency ω_m, cyclotron motion at ω_c, and an axial oscillation at ω_z (typically with $\omega_m << \omega_z << \omega_c$). Figure from Odom (2004).

which simultaneously changes the spin orientation and cyclotron level. After the cyclotron motion returns to thermal equilibrium, the continuous Stern-Gerlach effect is used to measure the spin state and determine the anomaly frequency ω_a. Since the cyclotron frequency is also simultaneously measured, this leads to a direct determination of a_e. The crucial feature of this experimental design is the exquisite precision that can be attained in these frequency measurements.

Hanneke, Hoogerheide, and Gabrielse (2008) represents the state-of-the-art in precision measurements of the electron's anomalous magnetic moment. A more detailed discussion of the experimental apparatus is found in (Hanneke, Hoogerheide, and Gabrielse 2011). The major advances in precision come from a few avenues. First, Hanneke, Hoogerheide, and Gabrielse (2008) replace the earlier hyperboloid geometry with a cylindrical Penning trap. The cylindrical cavity can be treated analytically, such that fringe effects are known and counteracted. This leads to a greater stability of the trapped electron, since "shifts of the electron's oscillation frequencies are avoided" (Hanneke, Hoogerheide, and Gabrielse 2011, p.3). Avoiding shifts in oscillation frequency within the cavity allows for a more precise measurement of $\frac{\omega_a}{\omega_c}$. Further, a surrounding electron plasma is used to determine frequency shifts of the cavity itself; knowing these cavity shifts eliminates a major source of uncertainty compared to previous measurements.

Next, Hanneke et al. employ quantum nondemolition measurements of the cyclotron and spin energy levels. A quantum nondemolition measurement leaves the measured state intact, so repeated measurements can be made without altering the state of the system. Formally, a quantum nondemolition measurement requires the Hamiltonian for the measured system—in this case the Penning trap—to commute with the Hamiltonian describing its interaction with the measuring system—here a one-particle self-excited oscillator. Additionally, the trapped electron "serves as its own magnetometer, allowing the accumulation of lineshape statistics over days" (Hanneke, Hoogerheide, and Gabrielse 2011, p. 1). The repeated measurements improve the accuracy of frequency measurements, and reveal that a major source of error remaining is the broadening of expected lineshapes over time.

All of these improvements lead to a measurement of the electron's magnetic moment to a precision of 0.28ppt:

$$a_e(\mathrm{HV08}) = 115\ 965\ 218\ 0.73(28) \times 10^{-12}. \tag{3}$$

Here the notation $HV08$ indicates the Harvard group's 2008 measurement. The physics of Penning traps does not depend in any close way on the details of QED. Non-relativistic quantum theory with classical electromagnetic fields is almost entirely sufficient to understand and control the single-electron stored in the Penning trap. Though the experiment is conducted as a test of QED, and therefore is carried out in the $\mathrm{QED_P}$ framework, the the Penning trap and measurement apparatus are not modelled within QED.

3.2 Predicting a_e and determining α

Aoyama et al. (2012; 2018) performed a calculation of theoretical contributions to the anomalous magnetic moment of the electron, up to the 10th order.[7] The primary novel contribution in these papers is the calculation of Feynman diagram amplitude contributions at the 10th order, as well as an improved precision calculation of 8th order terms. For the current state of precision measuerments, as discussed above, Aoyama et al. had to compute contributions to the anomalous magnetic moment that go beyond QED to include other parts of the standard model. The contributions to the anomalous magnetic moment of the electron can roughly be broken down additively as follows:

$$a_e(\text{theory}) = a_e(\text{QED}) + a_e(\text{Hadronic}) + a_e(\text{Electroweak}), \tag{4}$$

with $a_e(\text{QED})$ being the $\mathrm{QED_P}$ contribution, while $a_e(\text{Hadronic})$ the contribution from low energy quantum chromodynamics in the form of hadronic vacuum polarization and light-light hadronic scattering. Weak effects $a_e(\text{Electroweak})$ are calculated from one and two loop contributions to the muon $g-2$, suitably adapted to the electron.[8] The $\mathrm{QED_P}$ contribution can be calculated as shown in equation (2), replacing $F(\alpha)$ with $a_e(\text{QED})$ and A_n with $C_{ae}^{(2n)}$:

$$a_e(\text{QED}) = \sum_{n=0}^{\infty} C_{ae}^{(2n)} \left(\frac{\alpha}{\pi}\right)^n. \tag{5}$$

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[7] Their result is 10th order in the electric charge—the expansion is taken to $(\alpha/\pi)^5$.

[8] The separation is not purely additive, since $a_e(\text{Hadronic})$ will contain contributions from QED at higher orders, and $a_e(\text{Electroweak})$ will contain hadronic and QED corrections. However, the non QED corrections to a_e are not of sufficient precision for these nonlinearities to affect the prediction, and the separation still maps an important distinction between $\mathrm{QED_P}$ and $\mathrm{QED_{SM}}$ contributions.

Each $C_{ae}^{(2n)}$ can further be broken down into terms independent of lepton mass, terms depending on the electron mass (m_e), the muon mass (m_μ), and the tau mass (m_τ):

$$C_{ae}^{(2n)} = A_1^{(2n)} + A_2^{(2n)}(m_e) + A_3^{(2n)}(m_e/m_\mu) + A_4^{(2n)}(m_e/m_\tau) + A_5^{(2n)}(m_e/m_\mu, m_e/m_\tau). \tag{6}$$

The simplest of these terms are the set of $A_1^{(2n)}$, and the first three terms are known analytically. A_1^8 depends on contributions from 891 Feynman diagrams, while A_1^{10} depends on 12,672 vertex diagrams. These terms must be evaluated numerically—A_1^{10} requires grouping of diagrams into distinct families, running separate solvers on each group. Families of graphs share formal integral expressions, which allows one to reduce the total number N of integrals to evaluate numerically. This helps to reduce the overall numerical error, since the error associated with the sum of N intergals scales as $\sqrt{N}$. A further complication is that the integrals must be renormalized before being entered into numerical solvers. The bulk of the Aoyama, M. Hayakawa, et al. (2012) paper provides a detailed account of solving the largest set of diagrams contributing to A_1^{10}, while other terms are taken or slightly improved from previous work. This consists of writing down and (UV and IR) renormalizing the integrals, and describing the numerical solvers used.

Hadronic contributions to the anomalous magnetic moment can be broken down to contributions to vacuum polarization and light-light hadron scattering. Virtual hadrons carrying a net electric charge can contribute to a vacuum polarizing effect, introducing new diagrams from outside of pure QED. The first three orders of this contribution are included in a_e(Hadronic). Light-light scattering diagrams are predominantly mediated by electrons, though hadrons contribute to this interaction term as well. The electroweak term is solved analytically for one- and two-loop weak effects on the self-energy of the electron, and includes diagrams where W and Z boson contribute.

Aoyama, M. Hayakawa, et al. (2012) are cognizant of the need for an independent input value of α in order to determine the QED contribution to a_e.

> To compare the theoretical prediction with the measurement, we need the value of the fine-structure constant α determined by a method independent of [the anomalous magnetic moment]. The best α available at present is the one derived from the precise value of h/m_{Rb}, which is obtained by the measurement of the recoil velocity of rubidium atoms on an optical lattice. (p. 3)

Using the rubidium recoil input value of

$$\alpha^{-1}(Rb10) = 137.035\ 999\ 049(90)\ [0.66ppb], \tag{7}$$

the total combined contributions to a_e give a value of

$$a_e(\text{theory}) = 1\ 159\ 652\ 182.032(13)(12)(720) \times 10^{-12}, \tag{8}$$

where the first error term comes from numerical integration error of the 10th order QED_P contribution and the second error term is a combination of numerical integration and measurement errors for inputs to the hadronic contribution. The third (and largest) uncertainty is due to the experimental uncertainty in the fine-structure constant $\alpha(Rb10)$. The weak effects are calculated analytically, and therefore do not contribute to the theoretical error.

Looking at the uncertainties in $a_e(\text{theory})$ and $a_e(HV08)$, we see that the theoretical uncertainties are far smaller than the experimental uncertainties, with the largest source of uncertainty in $a_e(\text{theory})$ coming from the original input value of the fine-structure constant. Aoyama, M. Hayakawa, et al. (2012) make the same observation:

> The intrinsic theoretical uncertainty ($\sim 38 \times 10^{-15}$) of $a_e(\text{theory})$ is less than 1/20 of the uncertainty due to the fine-structure constant [Eq. (8)]. This means that a more precise value of α than [Eq. (8)] can be obtained assuming that QED and the standard model are valid and solving the equation $a_e(\text{theory}) = a_e(\text{experiment})$ for α. (p. 3)

This leads to a new value for α of:

$$\alpha^{-1}(a_e) = 137.035\ 999\ 1491(15)(14)(330), \tag{9}$$

where the uncertainties are due to the tenth-order QED_P prediction, the hadronic correction, and the experiment, respectively.

As mentioned above, the Hanneke, Hoogerheide, and Gabrielse (2008) experimental value of the anomalous magnetic moment is $a_e(\text{HV08}) = 1\ 159\ 652\ 180.73(28) \times 10^{-12}$, such that the agreement with $a_e(\text{theory})$ is very high:

$$a_e(\text{theory}) - a_e(\text{HV08}) = (1.30 \pm 0.77) \times 10^{-12}. \tag{10}$$

Agreement to a precision within the uncertainties of both the $a_e(\text{theory})$ and $a_e(HV08)$ *requires* that $a_e(\text{hadronic}) = 1.705 \times 10^{-12}$ and $a_e(\text{electroweak}) = 0.0297 \times 10^{-12}$ are included in the total calculation of the electron's anomalous magnetic moment. The contribution to a_e from hadronic and electroweak effects is $a_e(\text{Hadronic}) + a_e(\text{Electroweak}) = 1.735 \times 10^{-12}$.

The increased precision in measurement from Hanneke, Hoogerheide, and Gabrielse (2008) would have led to a slight discrepancy from a QED_P prediction. At this point, we already see the QED_P research program pointing to new physical details not included in its description of the electron, positron, photon, and their interactions. Additional factors from the other forces of the standard model are required to fully account for the best experimental value. Thus, we must shift from the testing framework of QED_P to QED_SM. The research program of precision testing in QED has led to the discovery that the electron's self-interaction contains significant interaction effects from outside pure QED.[9]

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[9] Of course, there are likely other ways to adjust the prediction $a_e(\text{theory})$ to reduce the

An alert reader might object at this point: why isn't the discrepancy between the pure QED value of a_e and $a_e(HV08)$ an anomaly that casts doubt on the entire testing framework? On our view this discrepancy requires a response, although by itself it does not suffice to reject QED_P for two distinct reasons. The first is the history of increasing convergence between measured and theoretically determined values of the anomalous magnetic moment, beginning at the inception of QED. The ever more precise agreement between higher-order expansions of a_e(theory) and higher precision methods for determining a_e(experiment) has led to a high degree of confirmation that QED_P accurately describes the relevant physics, at least to a very high level of approximation. But, second, we can assess the limitations of QED_P from the standpoint of the standard model. From this perspective, we see that real-world particle interactions are highly complex and nonlinear; one might naturally suspect that even relatively "clean" systems like the electron self-energy cannot be described with QED alone,[10] and that a shift to QED_SM may be required even for these elegantly designed experiments. The success in getting more and more precise agreement between theory and experiment with QED_P provides support to QED as correctly describing the dominant factors contributing to the anomalous magnetic moment of the electron. The presence of a discrepancy then indicates that the idealization of a pure QED system no longer holds, and one must move to a QED_SM testing framework. Since the discrepancy is resolved by introducing contributions from the strong and weak sectors of the standard model, we actually end up learning *more* about the nature of a_e. We learn that hadronic vacuum polarization, light-light scattering, and weak virtual processes make a measurable contribution to a_e. The discrepancy serves as an indicator of the limits of the highly idealized model, and the theoretical framework of the standard model gives us the additional physics needed.

The second interesting feature of Aoyama et al.'s (2012; 2018) work is that it serves two purposes: first, they predict a value of the anomalous magnetic moment of the electron to compare with the experimental results of Hanneke, Hoogerheide, and Gabrielse (2008). Second, the low theoretical uncertainty associated with their calculation coupled with the low experimental uncertainty of the experimental result allows for a new determination of the fine-structure constant. This was the most precise determination of α on record by a substantial margin until very recently; we will discuss a new measurement with comparable

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discrepancy, though most would be considered rather ad hoc. One could modify pure QED at this stage, though that option would alter the predictions for many other electromagnetic phenomena. Due to the rigidity of theories like QED, slight modifications would have far-reaching consequences. A further option, as we will discuss below, is to begin to include higher order terms in an effective field theory that goes slightly beyond pure QED and differs from the standard model. While this is the most promising of the alternatives, it is still a speculative change, and we believe should be saved for *after* known interactions are properly accounted for (i.e., if discrepancies persist for the QED_SM program).

[10] Pure QED describes only electrons, positrons, and photons, while we know at the very least that other electrically charged particles will factor into the Feynman diagrams for electron phenomena. The electron also participates in weak interactions, and one should therefore expect some weak effects to factor into a precise description of the standard model electron.

precision in the next section.

As mentioned above, $\alpha(a_e)$ is calculated by assuming a_e(experiment) is exact, and that the 10th order QED expansion—plus the other standard model factors—exhaust the relevant theoretical factors to include. So, rather than $\alpha(Rb10)$ being used as the empirical input to determine a_e(theory), one uses $a_e(HV08)$ as empirical input to calculate $\alpha(a_e)$. It is important to remember that this result is independent of the prediction of a_e(theory). The values of $\alpha(a_e)$ and a_e(theory) cannot both be precisely correct, though the agreement between a_e(theory) and the experimental value $a_e(HV08)$ gives license to the new predicted value of $\alpha(a_e)$, since agreement (within error) between the predicted and measured value of the anomalous magnetic moment makes plausible that the Aoyama et al. calculation captures all relevant physics within the precision of the Hanneke et al. experiment. This is an ongoing process, that continues in light of new measurements — such as a recent measurement using an alternative method that apparently diverges from this result (discussed below).

3.3 Discrepancies

Smith (2014) remarks that Newtonian celestial mechanics was free of widely recognized discrepancies for only 30 years of its long history. Although our discussion above has emphasized the successes of ongoing precision determinations of α, we would be remiss if we did not also acknowledge that there are, fortunately, discrepancies in low energy tests of the standard model. Here we will briefly mention two issues directly relevant to the discussion above.

The first regards the anomalous magnetic moment of the muon a_μ. As with the electron, the anomaly arises due to quantum corrections to the Dirac value of $g = 2$. Calculations similar to Aoyama et al.'s described above carry over to the determination of the QED_P contributions, but the contributions from a_μ(Electroweak) and a_μ(Hadronic) are significantly larger (Blum et al. 2013). The contribution of heavier virtual particles to the anomaly scales as the square of the mass, and the muon is much more massive than the electron: $\frac{m_\mu}{m_e} \approx 207$. (The lowest order hadronic contribuion to a_e is on the order of 1.5 parts per billion, compared to 60 parts per *million* for a_μ.) Although this means that challenging calculations of hadronic effects cannot be neglected in determining a_μ, it further implies that this measurement is sensitive to higher energy scales (up to the TeV range). Precision experiments of a_μ probe more of the standard model than measurements of a_e, and are also expected to have increased sensitivity to beyond standard model physics. Experiments to measure a_μ have to contend with its short half-life and much greater mass, and have so far have not reached the same level of precision as with the stable, lighter electron, with a_μ measured at .54 ppm precision (Bennett et al. 2004).

Current theoretical calculations have roughly the same level of uncertainty as the best experimental measurement of a_μ. Unlike the case of the electron, the results disagree at $\approx 3\sigma$ (Mohr, Newell, and Barry N. Taylor 2016, §V.B). A new experiment underway at Fermilab aims to reduce the uncertainty with respect to the Brookhaven results by a factor of 4, alongside further theoretical work aiming to extend and improve the standard model calculations. If we presume that the next generation results have the same central values as current results, with projected increases in precision, the discrepancy would be $7-8\sigma$ (Blum et al. 2013). Either experiment or theory may belie these expectations, obviously, and to the best of our knowledge the status of this discrepancy remains an important open question.[11]

Second, experimental measurements of atomic recoil have recently reached a level of precision slightly higher than that achieved for a_e.[12] Following seminal work by S. Chu, this line of research finds the ratio $\frac{h}{m_A}$, where h is Planck's constant and m_A is mass of an atom, by measuring the atom's recoil velocity upon absorbing a photon using a matter-wave interferometer. The value of α is then determined by the following equation:

$$\alpha^2 = \frac{2R_\infty}{c}\frac{m_A}{m_e}\frac{h}{m_A}, \tag{11}$$

where the Rydberg constant R_∞ and the ratio $\frac{m_A}{m_e}$ have both been measured independently with extremely high precision. Recent measurements take advantage of a huge improvement in precision measurements of $\frac{m_A}{m_e}$ (emphasized in Mohr, Newell, and Barry N. Taylor 2016), which is now known to better than 10^{-10} for some atoms. Based on measurements of the recoil of Cesium atoms, R. H. Parker et al. (2018) report a value of $\alpha^{-1}(Cs)$:

$$\alpha^{-1}(Cs) = 137.035\,999\,046(27) \tag{12}$$

The quoted total uncertainty is mostly due to statistical and systematic uncertainty associated with the measurement of $\frac{h}{m_A}$ (.20 ppb), with the remainder from uncertainty in R_∞ and mass ratios. This result, while consistent with earlier determinations of α^{-1} from atomic recoil measurements, diverges from the results based on the electron's gyromagnetic ratio by 2.5σ. These results clearly pose a challenge, forcing more careful evaluation of both experiments and QED itself. Discrepancies of this sort are sometimes resolved by identifying new sources of systematic error in one of the experiments. Gabrielse's group is pursuing a refined experimental design for measuring a_e at even higher precision (Gabrielse et al. 2019). If the discrepancy persists, it could signal that some subset of measurements does not provide a reliable determination of α at the stated level of precision. If so, this might indicate new physics playing a significant, thus far unaccounted for role in, for example, the electron's anomalous magnetic moment. Discrepancies play an important role in the process of precision testing, as they encourage more careful analysis of potential systematic error in experimental design. On the side of theory, discrepancies could signal new contributions that have yet to be accounted for, either within the

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[11] There have been proposals to account for a_μ with beyond standard model physics. An anonymous reviewer alerted us to Morishima, Futamase, and Shimizu (2018)'s proposal to resolve the anomaly by including gravitational effects. The proposal has been criticized, however, for mistakes in its treatment of gravity (Visser 2018).

[12] We thank George Smith for bringing this work to our attention.

QED_{SM} framework, or potentially beyond. One way to decide between competing determinations of α is by using independent precision tests, as we describe below.

4 Independent measurements of α: The quantum Hall effect

One might worry that, if all of the various precision determinations of $\alpha = e^2/\hbar c$ depend on QED-mediated calculations, then the constant increase in precision for determining α is merely a consistency check on QED. We have tried to show in the previous section that this is not the case. But over and above the convergence and increased precision offered by different theory-mediated measurements, as well as the multiple converging theoretical predictions, we have access to independent methods for determining α. These are not theory-independent methods, but rely on theories other than QED. The best independent measurements of α come from condensed matter effects; here we will discuss one such example, the quantum Hall effect.[13]

Most directly, one can use the independent measurements of the electric charge from classical electrodynamics, and $\hbar$ and c from non-relativistic quantum mechanics and relativity, respectively. This historically provided an initial value with which to start the process of ever increasing precision described above. Even this fact lends some support to the legitimacy of high-precision QED-mediated convergence. However, the limits of precision—most notable for classical electrodynamics—of rudimentary measurements mean that this provides only a rough initial starting point. Advances from condensed matter physics allow us to perform high-precision independent measurements of α. Though these tests don't match the extraordinary precision of the best QED-mediated measurements, consistency within error between QED- and condensed matter-mediated measurements provides further support that the standard model correctly describes the properties of the electron, and that these properties are stable regularities in nature. The most precise means of determining α outside of QED is via the quantum Hall effect.

The quantum Hall effect is a robust effect in condensed matter physics.[14] In (approximately) two-dimensional electron systems at low temperatures, the Hall conductance will undergo discrete transitions with an increasing magnetic field. The conductance $\sigma = \frac{I}{V_{Hall}} = k\frac{e}{h}$ is the inverse of the Hall voltage, where k can be a fraction with odd denominator (the fractional quantum Hall effect) or an integer (the integral quantum Hall effect). Only the integral quantum Hall effect is needed for precision measurements of α. The effect is robust because it appears to be insensitive to the particular type of material used, to the geometry

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[13] Historically, the ac Josephson effect was the first independent method to determine α, and it led to an important correction in the then accepted value of α (based on hyperfine splitting in hydrogen) (see Barry N Taylor, Langenberg, and W. H. Parker 1969).

[14] For more details on the quantum Hall effect, see Yennie (1987), Prange and Girvin (2012), and Tong (2016).

of the material's surface, or to the presence of impurities. As a result, the effect can be modeled in a rather simple, semiclassical approximation. Crucially, relativistic effects can be neglected, so the effect can be derived without any recourse to QED.

When a magnetic field is applied perpendicular to the 2D plane of the material, the electrons move in a cycloid pattern of radius $r = \frac{mv}{eB}$, with v the velocity of the electron, and B the magnetic field strength. Upon quantization, the allowed cyclotron orbits become discretized, with energy levels $E_n = \hbar\omega_C\,(n + 1/2)$, where $\omega_C = \frac{eB}{m}$ is the cyclotron frequency. In the quantum Hall effect, large magnetic fields are applied, so that ω_C is large. Each energy level—called a Landau level—is highly degenerate. The quantization of Hall conductance occurs when magnetic fields are sufficiently large that effectively all free electrons within the material occupy a single Landau level. At this point, the material's resistivity is attributable to the resistivity associated with a single Landau level, making high-precision measurements of resistivity as a function of B possible. This is why materials exhibiting the QHE are taken to be in macroscopic quantum states. The high degeneracy of the Landau level at high B and the low T used to suppress thermal fluctuations effectively make all of free electrons behave in sync as a single quantum state.

In the regime relevant for the quantum Hall effect, the conductance σ is measured as a function of B, and a plot of the resistance $\rho = 1/\sigma$ shows a stepwise increase with magnetic field strength, while plateaus in the resistance increase in width for higher magnetic fields. The precision with which the differences in resistance at each plateau can be determined indicate that k is an integer to a precision of approximately 1ppb, which leads to a highly accurate determination of the ratio e/h at low energies. Since $\alpha = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar c}$, while c and ϵ_0 are exactly defined quantities in the CODATA tables, the ratio determined through precision tests of the quantum Hall effect can be used to calculate α. Given the scaling behaviour of coupling "constants", one does not expect e, and therefore α, to be constant at all energy scales. The low energy precision tests of α should therefore give different results from high-energy tests—the β-function for QED indicates that α should increase with increasing energy. According to the particle data group, $\alpha \approx 1/128$ when $Q^2 = m_W^2$, and the precision value quoted is for $Q^2 = 0$.[15]

In practice, measurements of the quantum Hall effect are made to precisely determine the von Klitzing constant $R_K = \frac{h}{e^2}$, as this is a phenomenological standard for the fundamental unit of resistivity in materials. Experimentalists seek precision in this measurement for reasons of metrology, independent of the measurements of α from QED (cf. Trapon et al. 2003). However, the increased precision on R_K allows for increased precision in determining $\alpha = (4\pi\epsilon_0 c R_K)^{-1}$.

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[15] The OPAL collaboration running out of the LEP at CERN was conducted to measure running coupling of QED and QCD. They have found (OPAL-collaboration et al. 2006) conclusive evidence that α increases with centre of mass energy. This is where the CODATA value of $\alpha(Q^2 = M_W^2) \approx 1/128$ comes from.

The current state-of-the-art gives:

$$\alpha^{-1}(\mathrm{QHE}) = 137.035\,997\,9(32), \tag{13}$$

which—though not in exact agreement with Eq. 9—agrees to five decimal places.

The quantum Hall effect provides an important QED-independent measurement of α, and is one of the most precise means for measuring α—aside from the measurement of a_e, atom recoil, and indirect calculation from the neutron Compton wavelength. Once consistency has been established between values measured using QED and independent tests like the quantum Hall effect, the worry about circularity from using a QED-mediated result to test QED is assuaged, and one can treat these precision measurements as direct tests comparing theory to experiment.

5 Using precision tests to go beyond the standard model

The predominant view of QED—and the rest of the standard model—is that it is an effective field theory. What this means is that it is an approximation to a more fundamental underlying theory, and that under suitable approximations, the effective theory can be found within the more fundamental theory. An effective *field* theory is just an effective theory that employs fields in its effective description. In most effective field theories—such as condensed matter quantum field theory and hydrodynamics—the fields are usually interpreted as a continuum approximation of the entities in the underlying theory—atoms and molecules, respectively. For the standard model, the fundamental underlying theory is as yet unknown. If QED and the standard model are simply effective theories, and therefore ultimately at best approximations, why would physicists assume their validity when designing and conducting precision tests of the electron's properties?

$\mathrm{QED_P}$ and $\mathrm{QED_{SM}}$ form frameworks in which research programs of precision testing can be carried out. As Smith (2014) notes regarding the Newtonian framework, which set a research program for astronomers for over 200 years:

> [T]he primary question astronomers addressed when they compared calculations with observations is, What, if any, further forces are at work? The preoccupation of their research has not been with testing the theory of gravity, but with identifying further forces at work. To this end, their research presupposes the theory of gravity—or, as I prefer to express it, their research is predicated on the theory of gravity. (p. 266)

In much the same way, the back and forth process of measuring and determining from theory the anomalous magnetic moment—with ever more precision at each iteration—has been predicated on $\mathrm{QED_P}$, though currently thought to be an approximation to $\mathrm{QED}_{\mathrm{SM}}$ and the standard model more broadly. Our emphasis on the constitutive role theory plays in this reasoning may prompt an objection: to what extent can we still endorse the conclusions of this iterative process if we no longer accept Newtonian gravity, or pure QED? This depends on whether there is sufficient continuity between frameworks to ensure that the inferences to further forces, or further physical details, still hold. One quite specific form of this concern regards the interpretation of α itself: how does the quantity introduced by Sommerfeld relate to the coupling constant in the standard model Lagrangian? Our answer appeals to continuity of application: the application of $\mathrm{QED}_{\mathrm{SM}}$ includes, as an approximation valid in a limited domain, the application of Sommerfeld's non-relativistic theory. Generalizing from this point, we endorse Bain and Norton (2001)'s argument that there is a great deal of continuity through the changing theories of the electron over the last century. All we require is that the relations used as an essential part of the testing framework can be treated as good approximations, within a restricted domain, to relations that hold in the successor theory. We can then assess whether the identification of a discrepancy falls outside this domain. For example, in the case of celestial mechanics, the Newtonian analysis of the impact of Neptune on Uranus's orbit is presumably an extremely good approximation to a general relativistic treatment.[16]

As noted in Section 2, there are complications to viewing QED and Newtonian gravity as exactly analogous. First, discrepancies between measured and predicted values of the anomalous magnetic moment are not necessarily a sign of new forces at work. Due to the fact that QED is described in terms of an asymptotic series, at some (unknown) point higher-order contributions to the expansion will diverge from the "true" value. We cannot use perturbation theory to get arbitrarily close to some exact value predicted by the theory; such a value would require a more rigorous formulation of QED, to which the generating functional provides an approximation. Given the current structure of QED, there is no "true" underlying value it predicts, to which the generating functional is an approximation.[17] While complex approximations used to make predictions for n-body systems in Newtonian gravity could be asymptotic series, in principle the Newtonian theory gives some complicated but exact expression for the evolution of the system. A better approximation scheme might not have the same fault, and since the "true" equation is known, bounds can be placed

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[16] We say presumably because we are not aware of a fully relativistic treatment, which would be extremely challenging computationally; in practice, relativistic effects are typically included as corrections to Newtonian calculations.

[17] By this we mean that QED is not known to be Borel summable, such that its generating functional is thought to be asymptotic, but the functional to which it is asymptotic is unknown. One may take something like lattice QED, for example, to be the exact description of QED; in that case, some nonperturbative effects would be considered exact. In lattice QED, the cutoff in momentum is an integral part of QED as a dynamical model, and continuum limits are introduced for convenience at the end of a calculation. However, for predictions involving expansions in α, perturbation theory is still essential. If one prioritizes the continuum limit as essential to QED, then the fact remains that these perturbative expansions are perturbative to some unknown underlying value. We thank David Wallace for bringing up this possibility as a suggestion.

on the amount of error expected from an asymptotic approximation scheme.

Second, for Newtonian celestial mechanics, the further forces at work are (almost always) gravitational forces. The deviations come primarily from many-body interactions, or from changes to the mass distribution in the solar system.[18] These effects may be impossible to calculate analytically, and so require novel approximation procedures to include, but the working hypothesis is almost always that one has not taken into account the full details of other gravitating bodies. As we have seen for the anomalous magnetic moment, discrepancies between the measured value and the state-of-the-art QED_P required additional terms involving the strong and weak forces, and therefore a shift to a QED_SM framework. So the methods of reasoning are slightly more complex in the case of the standard model.

Finally, Smith (2014) makes much of the robustness of the new forces at work. For example, the hypothesis that Neptune existed and explained the anomalies in Uranus' orbit had to be verified through the observation of Neptune itself, and other effects it has on the solar system. This is a crucial aspect of the success of Newtonian celestial mechanics, and these causal dependencies survived the transition from Newtonian gravity to general relativity. Smith argues that cases like this license one to make causal claims about the dominant factors in the dynamics of the solar system, and that these are more robust than the Newtonian equations of motion used to determine them. Precision testing of QED lacks similarly concrete sources of new forces. In quantum field theory, the terms in a perturbative expansion are simply parts of the overall interaction added together. Despite the visual aid of Feynman diagrams, one should not think of each diagram as representing an actual interaction taking place, summed together to determine the totality of interactions affecting the electron. What we have learned is that terms of higher order have significant observational consequences in at least some QED phenomena. The diagrams will have a similar contribution in other QED processes, and the numerical value determined from Aoyama, M. Hayakawa, et al. (2012) could be used in calculating these other effects. As we discuss below, there is another sense in which the framework of effective field theory might be robust through the working out of precision effects within the standard model. Despite these differences, we are similarly licensed in making claims about the dominant causal factors in electromagnetic systems, and can be confident that these are more robust than the QED equations used to derive them.

Precision testing of QED_SM exhibits the same predication upon the standard model that precision tests of astronomy did for Newtonian gravity. Importantly for the effective field theory view of the standard model, presupposing Newtonian gravity did not impede the development of general relativity. As Smith (2014) notes,

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[18] They also include effects that arise due to modeling solar system bodies as extended bodies rather than point masses, which is particularly important in lunar theory, along with several other subtle effects. In a few cases, "non-gravitational" effects such as the slowing of the Earth's rotation due to tidal friction have to be taken into account.

> Strikingly, the transition from Newtonian to Einsteinian gravity, as a matter of historical fact, left all the previously identified details of our solar system that make a difference and the differences they were recognized as making intact. In other words, the details that make a difference in our solar system and the differences they make proved more robust in the transition to Einsteinian theory than the Newtonian theory that had provided the basis for identifying them. This collection of difference-making details therefore has the strongest claim to knowledge produced by the two centuries of research predicated on Newtonian theory. But Newtonian theory also has a claim to knowledge, namely as a theory that, while holding only approximately over a restricted domain, still was adequate to establish many details that make a difference and the differences they make within that domain. (p.266)

In this way, the effective field theory view of the standard model is not simply compatible with experimental work predicated on the standard model, but is necessary for precision testing. But, more than this, precision testing can be our best guide to breakdowns in the standard model, and may indicate discrepancies to be remedied by some "beyond the standard model" theory. Small deviations from theoretical expectations become more significant with increased measurement precision, as in the muon discrepancy and α tension discussed in Section 3.3.

Consider again the case of Newtonian gravity and general relativity. One crucial prediction[19] for Einstein was the remaining $43''$/century of Mercury's perihelion precession. Over the previous century or so, astronomers were able to come up with more and more detailed models of Mercury's orbit, and were able to account for much of the observed $575''$/century precession. Most of this precession is due to gravitational "tugs" from other solar system bodies, predominantly Jupiter. However, there were repeated failures to account for the further $43''$ discrepancy, and this signalled that perhaps some modification to the inverse square law was required. Were it not for the work of successive additions of additional forces at work highlighting the missing $43''$/century, it is likely that Einstein would not have had this crucial piece of evidence for his new theory. First, the lack of attention to the anomaly would have made it less likely that Einstein would have placed any significance on resolving it with general relativity. Second, even if Einstein calculated Mercury's orbit with

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[19] Some may be uncomfortable using with the term "prediction" to describe this episode, since the anomalous perihelion precession was known before Einstein derived it from the weak field limit of general relativity (Einstein 1915). Many distinguish between predictions, which occur before an effect is known or measured, and retrodictions, which occur after. We believe that what is epistemically relevant for a prediction is the independence of the construction of the theoretical apparatus from the "predicted" phenomenon. In this case, Einstein constructed the general theory of relativity without factoring in the anomalous precession of Mercury's perihelion. So the fact that Einstein's derivation made up for the missing factors counts as a prediction on this reading. However, this is not a point that is essential to what follows. The reader can substitute "retrodiction" for "prediction" in the text here without altering our point.

general relativity, the $43''$/century makes up only a tiny fraction of the observed precession. As Einstein put it in a letter to Sommerfeld, "Here we are helped by the pedantic precision of astronomy, which I often secretly poked fun at" (Schulmann et al. 1998, Volume 8, Doc. 161). The fact that general relativity predicts *some* precession is very different from predicting the amount needed to close the gap between predicted and observed precession after over a century of precision testing.

In much the same way, precision tests of $\mathrm{QED_P}$ and $\mathrm{QED_{SM}}$ may be crucial for testing new physics beyond the standard model. As we have seen with the case of the anomalous magnetic moment of the electron, precision testing has been a constant process of refining both observations and predictions. The majority of these tests are predicated upon QED, since the quantities they aim to measure are made physically significant only in the context of QED. Predictions from the theoretical side are constantly taking into account finer and finer details of QED interactions in order to derive more precise results. So far theory has been able to stay in precise agreement with observation, though the results of Aoyama et al. (2012; 2018) have marked an important step in that forces *outside* of $\mathrm{QED_P}$ are now required to maintain the agreement.[20] QED alone is no longer sufficient to account for the total known self-interaction of the electron. At this stage, known interactions (low-energy weak and hadronic effects) make up the difference. The ideal situation for those working on theories involving new physics would be a persistent discrepancy to emerge in the next round of precision tests—one that could not be remedied by including effects from the weak or strong forces.

The hope for new discrepancies from the standard model can be made more abstractly in the language of effective field theory. As Weinberg (1979) notes, the effective field theory program can be thought of as a recipe for writing effective Lagrangians, including a built-in cutoff scale at which the effective theory breaks down. The recipe is simple:

> . . . if one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles. (Weinberg 1979, p.329)

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[20] Nobel Prize winner Hans Dehmelt (1990)—one of the pioneers of precision measurements of electrons using Penning traps—believed that the fact that $g_{\mathrm{electron}} > g_{\mathrm{Dirac}}$ indicated that the electron had structure beyond that of a point particle. This is one very literal way to think of how precision testing illuminates the structure of the electron. In one sense, a composite electron would be a major departure from the assumed ontology of the standard model, since the electron is treated as a fundamental, rather than composite, field. However, if the anomalous magnetic moment is evidence for a composite electron, then the standard model already implies that the electron is structured. While slightly orthogonal to the discussion in the main text, this could be another important sense in which precision testing would point to new physics.

Importantly, many of these terms will not be renormalizable; this is where the effective field theory methodology comes into play. If we expect this Lagrangian to describe an effective dynamics, valid only up to some energy scale Λ, then terms in the Lagrangian with mass dimension higher than four will be suppressed by powers of Λ. Relating this back to the standard model, one can consider effective theories beyond the standard model by writing effective Lagrangians for sectors of the standard model. One in principle includes all higher-order terms consistent with the symmetries of that sector, though terms with higher mass dimension will be suppressed by higher powers of Λ. Precision testing of the standard model will then place upper bounds on the magnitude of new nonrenormalizable terms added in this effective Lagrangian. For the case of the anomalous magnetic moment discussed above, the close agreement between the $\mathrm{QED}_{\mathrm{SM}}$ predicted value and the measured value implies that an effective extension of $\mathrm{QED}_{\mathrm{SM}}$ could only contain very small contributions from nonrenormalizable terms, much like the contributions from the standard model were very small modifications to $\mathrm{QED}_{\mathrm{P}}$. If the additional terms are too large, then the agreement between theory and experiment would be spoiled.[21] Precision tests of other QED effects ensure that new effective terms aren't carefully constructed to have negligible effect on one set of observables, while having a major effect on others. In reasonable effective field theories, a small effect added into a beyond $\mathrm{QED}_{\mathrm{SM}}$ effective theory to modify the anomalous magnetic moment would factor in elsewhere, potentially creating discrepancies between the new theory and other measured effects. Though we have largely focused on one test of QED, multiple lines of evidence—both low-energy precision evidence and high-energy effects—provide essential constraints on future theories.

Though significant deviation from the standard model at low energies is only an outside possibility, the prospects look dim that more direct, traditional methods in particle physics will reveal new physics any time soon. Many physicists fully expected evidence of naturalness at the LHC when the Higgs was discovered—in the form of new particles or evidence of new forces.[22] However, up to this point, no new physics has been discovered. The simple fundamental scalar Higgs seems to be the best supported by data from the LHC, meaning that the discovery of the Higgs has not provided any insight into physics beyond the standard model.[23] Further, guided searches using simplified models

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[21] It is interesting to note that the best lower bounds on the value of Λ come from high-energy lepton collisions, rather than low-energy precision tests. At high energies, the collisions depend on the full electroweak sector of the standard model. Close agreement between standard model calculations and experiment indicates that $\Lambda > 500 GeV$, (cf. Kinoshita (1990, Sec. 5.3.3), Sapirstein (2006, Sec. 27.3)) which provides an upper bound on the magnitude of reasonable corrections. Here we see effects from very different energy scales working together to constrain modifications of QED.

[22] Giudice (2017) is a prominent example of this sort of thinking. As the head of the theoretical team at CERN, his thoughts on the role of naturalness have evolved heavily since the dawn of the LHC (Giudice 2008).

[23] Incidentally, one can also calculate the Higgs contribution to a_e, as in (cf. Peskin and Schroeder 2018, Problem 6.3), $a_e(\mathrm{Higgs}) \approx \mathcal{O}(10^{-24})$. Though this is still within the standard model, and its effect is too small be currently observable, a non-scalar Higgs would change the contribution and this in principle another way to use precision testing to find evidence of

have not found any new particles beyond the standard model (McCoy and Massimi 2018). The LHC is operating at its maximum energies, so the lack of new physics discovered is concerning. Moreover, the LHC is entering a phase of testing, after years in the previous "search" phase. During the search phase, experiments were focused on finding evidence of the existence of new particles. Despite a few anomalies, search only found evidence strong enough to claim a discovery for the Higgs boson. The testing phase primarily involves producing large quantities of Higgs bosons, to better determine their properties.[24] For this purpose, total centre of mass energy will be sacrificed for beam intensity near the Higgs threshold. By using an effective field theory framework, theorists hope to fit the new precision Higgs data into consistent field theories that extend beyond the standard model. Brivio and Trott (2019) outline some of this work, including two extensions to the standard model in which to accommodate new Higgs data. Though this new avenue of research is promising, it is a departure from the direct accelerator tests conducted for the last seventy years. If some new theory predicts new physics at an energy scale above the current threshold of the LHC, the only direct testing method would involve building a new, bigger accelerator. This would be an expensive, time-consuming undertaking. Given that the Superconducting Supercollider was rejected by US Congress, and the missing Higgs boson has been discovered, most high-energy physicists are pessimistic about the odds of a new accelerator being funded. Even if funding is approved, a new accelerator would take years to complete; for the time being, indirect methods of testing are the only window into physics beyond the standard model.

QED is a prime candidate for precision testing for a few reasons. First, most of the easily manipulable stable particles interact primarily via the electromagnetic force, and therefore may be treated more easily as pure QED systems. Electrons are particularly useful for study, as they do not experience the strong force, and are stable under weak interactions. Second, the electromagnetic force is the weakest of the three forces described by the standard model,[25] and is best suited to perturbative expansion. The process of refining predictions outlined in Section 2 depends on an increased precision coming from adding new terms from the perturbative expansion in powers of α. For nonpertubartive effects in the standard model—most notably in the low-energy quark regime—precision is possible in experimental detection, but difficult from the side of theory.[26]

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physics beyond the standard model.

[24] Though the precision here will ultimately be far lower than precision testing of the properties of the electron, the goal of testing the properties of the Higgs boson is to better understand what sort of particle it is. One hope is that increased precision on its properties will reveal that it is incompatible with the simple scalar description provided by the standard model; this would hint at new physics to be explained by future theory.

[25] At least when ordered by coupling constant; timescales associated with weak decays can often be shorter than electromagnetic decays.

[26] A notable "precision test" of the standard model currently being conducted by numerous groups is the search for evidence of nonzero neutrino masses. Mixing of solar neutrinos suggests that at least 2 of the three neutrino flavours from the standard model have nonzero mass (Fukuda et al. 1998; Battye and Moss 2014). Neutrinos are currently treated as massless within the standard model, but models that factor in nonzero mass have been constructed

There is precedent elsewhere in physics for using precision tests as a means to place tight bounds on the deviation of parameters predicted by current theory. Precision testing of general relativity was sought as early as the 1960s, and precise deviations from the theory were formalized in the parameterized post-Newtonian framework (Will 1971). This framework represents a limitation of general relativity to the weak gravitational (Newtonian) regime, and elevates certain key quantities that would parameterize deviations from general relativity into variables. Given this framework, one can characterize alternative theories of gravity by the value they give to these variables in a "theory-space" of post-Newtonian theories. Precision measurements come in to place bounds on the variables characterizing the post-Newtonian theory space. A similar formalism has been developed more recently to characterize deviations from the ΛCDM model of cosmology, and the Friedmann-Lemaitre-Robertson-Walker spacetime on which it is predicated (Baker, Ferreira, and Skordis 2013). The benefits of exploring—and ruling out—large areas of theory space with precision testing are clear. One can constrain future theories that go beyond our current best models, even in the absence of concrete proposals. Further, when new models are proposed, they can quickly be parameterized to fit within the theory-space, and the parameters from the reduced theory compared to experimental bounds.

One should not overstate the power of this form of testing, however. Any so-called "model-independent" framework—such as the parameterized post-Newtonian framework—must still make substantial assumptions in order to have any quantitative power. The parameters chosen will encode the community's expectations regarding the ways in which future theory will deviate from current theory. For the effective field theory framework described above, one assumes the principles of analyticity, unitarity, and cluster decomposition, as well as any other hidden assumptions implicit in the quantum field theoretical framework. This sort of project is not exempt from Stanford's (2006) problem of unconceived alternatives. The assumptions made in constructing the parameterized theory space might still miss important alternatives. Fundamental changes to the concepts of the current theory—the standard model, general relativity, or the ΛCDM model of cosmology—are unlikely to be captured by these rather conservative extensions of the current theoretical framework. Simply replacing select constants in the standard model with unconstrained parameters does little to alter the conceptual framework currently in place. Further, the map from possible future theories to the parameterized theory space could be many-one, and there is no natural measure on the latter. This means that there is no well-defined sense in which one is ruling out large domains in the parameterized space, or that doing so would entail that realistic candidate theories are thereby heavily constrained.

That said, placing bounds on deviations from current theoretical expectations is a highly systematic method of constraining future theory, especially when little is known about the contours of that future theory. Whatever shape

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(Gonzalez-Garcia and Maltoni 2008; Ma 1998). Though outside the scope of this paper, these precision tests could also point to new physics beyond the standard model.

the future theory takes, the history of theory change suggests that it should be possible to limit its domain to the appropriate regime to compare the value of, say, the anomalous magnetic moment of the electron to that predicted by the standard model. Though the issues mentioned in the previous paragraph cannot be avoided, we must acknowledge that theory construction is never infallible. Until a new theory is constructed whose restriction to low energies does not fall meaningfully outside of the standard model range in parameter space, we cannot say that important alternatives are missed in theory space. Precision testing has an important role to play in the current landscape of theoretical physics, and precision testing of QED_{SM} is the current gold standard in testing the standard model.

6 Conclusions

Precision testing of QED is a subtle process that relies on a mutual interaction between experimentalists and theorists. The fine-structure constant α is the key input needed for making predictions in QED, and can be experimentally determined in various ways. We have discussed the anomalous magnetic moment of the electron—arguably the most famous and most precise QED prediction—and the quantum Hall effect as QED-mediated and independent methods of measuring α, respectively.

We should acknowledge, in closing, one aspect of the impact of this line of work that we do not have the space to explore here.[27] Planck identified the precision measurement of interlinked constants appearing in the laws governing microscopic phenomena, and the introduction of natural units based on them, as important targets for further research in 1900. The equations above link α with several other fundamental constants $(h, e, c, R_\infty, R_K, ...)$, so theory-mediated precision determinations of α obviously have ramifications for the other fundamental constants (except for Newton's constant, G). The late 1960s saw substantial progress in several different experimental techniques designed to measure α, summarized in the influential review (Barry N Taylor, Langenberg, and W. H. Parker 1969). Along with this progress in metrology and experimental design was a striking emphasis on the value of using QED independent methods of determining α, in conjunction with those that do depend on core assumptions of the theory, to test QED. The periodic reviews stating the recommended values of the fundamental constants published by the CODATA group often emphasize the strong evidence in favor of QED based on such consistent determinations of the constants (see, e.g., Mohr, Newell, and Barry N. Taylor 2016). The epistemic questions that arise in this interplay between fundamental physics and metrology deserve further study.

Returning to our main theme, as more and more precise measurements become possible, more sophisticated techniques are required to extract more detailed predictions from the generating functional of QED. Supposing that QED as part of the standard model is in agreement with experimental results within

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[27] We thank an anonymous reviewer for helpful comments that prompted this paragraph.

the margin of error, the possible deviations from the standard model are constrained. This is useful in the process of constructing theories that go beyond the standard model, since these will ideally predict deviations from certain expected quantities within the standard model. A combination of high-energy and low-energy tests constrain the magnitude of new parameters that could be added to a beyond standard model effective theory.

Though both pure QED the standard model are expected to be approximations to some deeper theory, physicists assume their validity in the process of precision testing. Faith in an ultimately approximate theory is justified on multiple grounds. First, the properties thus determined remain relatively stable, and survive theory change to a high degree of approximation. Bain and Norton (2001) detail the accumulation of knowledge in the history of theories of the election; like the charge to mass ratio and the quantized charge of the electron, the value of the anomalous magnetic moment will remain a well-defined property added to the growing core of electronic properties. Next, as we have argued extensively above, the standard model determines the dominant physical factors that affect the self-energy of the electron. By determining as precisely and exhaustively as possible the magnitude of standard model effects, persistent deviations will provide hints for future theories.

One can see the precedent for using precision tests of theories to aid future theory construction throughout physics. We discussed the example of the precession of the perihelion of Mercury in some detail, and mentioned more contemporary examples from cosmology. But there is value to the knowledge generated through precision testing, beyond its use for constructing future theories. To paraphrase Smith (2014), the standard model has so far proven adequate to establish details discovered in the process of precision testing QED. These details make a difference to our measurements, and the standard model establishes the nature of the differences they make.

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